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# Laplace-transform-based method to calculate back-reflected radiance from an isotropically scattering half-space

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**Abstract.** We present a method to determine the back-reflected radiance from an isotropically scattering half-space with matched boundary. This method has the advantage that it leads very quickly to the relevant equations, the numerical solution of which is also quite easy. Essentially, the method is derived from a mathematical criterion that effectively forbids the existence of solutions to the transport equation which grow exponentially as one moves away from the surface and deeper into the medium. Preliminary calculations for infinitely wide beams yield results which agree very well with what is found in the literature.

## 1. Introduction

The case of a pencil beam incident on a half-space, also called the searchlight problem, has been dealt with many times, both exactly [1–5], in the diffusion approximation [6–8], by the Monte Carlo method [9] and by random walk theory [10]. Exact methods tend to be lengthy and complicated, for instance, Rybicki [4] needs some 20 pages of analysis to obtain his numerical results in Fourier space, after which a Fourier inversion still has to be effected. On the other hand, Hoenders has outlined a method from which the pertinent equations follow immediately and which has the added advantage that all the equations to be solved are linear. Here, his ideas are worked out for the simple case of a half-space containing isotropic scatterers, illuminated by a collimated beam of infinite width. In the conclusion we will explain that finite beams as well as anisotropic scattering can be handled within the framework of the present method.

## 2. Method

We start by considering an isotropically scattering half-space with index-matched boundary. Inside the medium, the time-independent transport equation applies

$$\left( \Omega_x \frac{\partial}{\partial x} + \Omega_y \frac{\partial}{\partial y} + \Omega_z \frac{\partial}{\partial z} \right) L(\mathbf{x}, \boldsymbol{\Omega}) + L(\mathbf{x}, \boldsymbol{\Omega}) = \frac{a}{4\pi} \int_{4\pi} L(\mathbf{x}, \boldsymbol{\Omega}') d\boldsymbol{\Omega}'. \quad (1)$$

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Here  $L(\mathbf{x}, \boldsymbol{\Omega})$  is the radiance at position  $\mathbf{x}$  in direction  $\boldsymbol{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ ,  $a$  is the albedo and  $z$  measures the depth in the medium. Lengths are measured in units of  $\mu_t^{-1}$ , i.e. the total mean free path, due to both scattering and absorption.

Next, we take the Laplace transform with respect to the variable  $z$  and, following Elliot [11], the Fourier transform with respect to the variables  $x$  and  $y$ . Defining the quantities  $\mathcal{L}$ ,  $\mathbf{k}$ ,  $\boldsymbol{\omega}$  and  $\mu$  to be

$$\mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu) = \int_0^\infty dz \int_{-\infty}^\infty dy \int_{-\infty}^\infty dx L(\mathbf{x}, \boldsymbol{\Omega}) \exp[-ik_x x - ik_y y - sz] \quad (2)$$

$$\mathbf{k} = (k_x, k_y) \quad \mu = \Omega_z \quad \boldsymbol{\omega} = (\Omega_x, \Omega_y) \quad (3)$$

we can express the result of the combined Fourier–Laplace transform as

$$\begin{aligned} \mu[s\mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu) - L_0(\mathbf{k}, \boldsymbol{\Omega})] + (i\mathbf{k} \cdot \boldsymbol{\omega}) \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu) + \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu) \\ = \frac{a}{4\pi} \int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}'. \end{aligned} \quad (4)$$

Here  $L_0(\mathbf{k}, \boldsymbol{\Omega})$  denotes the Fourier transform of the radiance at the boundary surface  $z = 0$ , i.e.

$$L_0(\mathbf{k}, \boldsymbol{\Omega}) \equiv \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy L(x, y, z=0, \boldsymbol{\Omega}) \exp[-ik_x x - ik_y y]. \quad (5)$$

Solving (4) for  $\mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu)$  yields

$$\mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu) = [\mu s + i\mathbf{k} \cdot \boldsymbol{\omega} + 1]^{-1} \left\{ \frac{a}{4\pi} \int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}' + \mu L_0(\mathbf{k}, \boldsymbol{\Omega}) \right\}. \quad (6)$$

Now, we invoke a criterion stating that the Laplace transform  $\mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}, \mu)$  cannot have singularities for  $\text{Re } s > 0$ , since in real space, these would give solutions that grow exponentially for  $z \rightarrow \infty$  [12]. Hence, the pole in equation (6) must be compensated for, that is, we have

$$\frac{a}{4\pi} \int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}' + \mu L_0(\mathbf{k}, \boldsymbol{\Omega}) = 0 \quad \text{for } s = -\frac{1}{\mu}(i\mathbf{k} \cdot \boldsymbol{\omega} + 1). \quad (7)$$

This equation has been obtained previously by Rybicki [4]. Because the above-mentioned criterion is limited to values of  $s$  for which  $\text{Re } s > 0$ , equation (7) is valid only for  $\mu < 0$ . We can eliminate the unknown Laplace transform in equation (7) by integrating equation (6) with respect to  $\boldsymbol{\Omega}$  and solving for  $\int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}'$ . Thus, upon integrating (6), we find

$$\begin{aligned} \int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}' &= \frac{a}{4\pi} \int_{4\pi} \frac{d\boldsymbol{\Omega}}{\mu s + i\mathbf{k} \cdot \boldsymbol{\omega} + 1} \int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}' \\ &+ \int_{4\pi} \frac{\mu L_0(\mathbf{k}, \boldsymbol{\Omega})}{\mu s + i\mathbf{k} \cdot \boldsymbol{\omega} + 1} d\boldsymbol{\Omega} \end{aligned} \quad (8)$$

and solving for  $\int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}'$  (which is the Fourier–Laplace transform of the fluence rate)

$$\int_{4\pi} \mathcal{L}(\mathbf{k}, s, \boldsymbol{\omega}', \mu') d\boldsymbol{\Omega}' = \left[ 1 - \frac{a}{4\pi} \int_{4\pi} \frac{d\boldsymbol{\Omega}'}{\mu' s + i\mathbf{k} \cdot \boldsymbol{\omega}' + 1} \right]^{-1} \int_{4\pi} \frac{\mu' L_0(\mathbf{k}, \boldsymbol{\Omega}')}{\mu' s + i\mathbf{k} \cdot \boldsymbol{\omega}' + 1} d\boldsymbol{\Omega}'. \quad (9)$$

Using this result in equation (7), we get

$$\frac{a}{4\pi} \int_{4\pi} \frac{\mu' L_0(\mathbf{k}, \boldsymbol{\Omega}')}{\mu' s + i\mathbf{k} \cdot \boldsymbol{\omega}' + 1} d\boldsymbol{\Omega}' + \mu L_0(\mathbf{k}, \boldsymbol{\Omega}) - \frac{a}{4\pi} \mu L_0(\mathbf{k}, \boldsymbol{\Omega}) \int_{4\pi} \frac{d\boldsymbol{\Omega}'}{\mu' s + i\mathbf{k} \cdot \boldsymbol{\omega}' + 1} = 0. \quad (10)$$

Substituting the value  $s = -(\mathbf{i}\mathbf{k} \cdot \boldsymbol{\omega} + 1)/\mu$ , from equation (7) and multiplying out the denominators, we find

$$\frac{a}{4\pi} \int_{4\pi} \frac{\mu' L_0(\mathbf{k}, \boldsymbol{\Omega}')}{\mathbf{i}\mathbf{k} \cdot (\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + (\mu - \mu')} d\boldsymbol{\Omega}' + L_0(\mathbf{k}, \boldsymbol{\Omega}) - \frac{a}{4\pi} \mu L_0(\mathbf{k}, \boldsymbol{\Omega}) \int_{4\pi} \frac{d\boldsymbol{\Omega}'}{\mathbf{i}\mathbf{k} \cdot (\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + (\mu - \mu')} = 0 \quad \mu < 0. \quad (11)$$

Notice that equation (11) only contains the quantity  $L_0(\mathbf{k}, \boldsymbol{\Omega})$ , which is related to the incoming and outgoing radiance at the surface, that is, all quantities belonging to the radiance inside the medium have been eliminated. Now denoting  $L_0(\mathbf{k}, \boldsymbol{\Omega})$  by  $L_+(\mathbf{k}, \boldsymbol{\Omega})$  for  $\mu' = \boldsymbol{\Omega}'_z > 0$  and by  $L_-(\mathbf{k}, \boldsymbol{\Omega})$  for  $\mu' = \boldsymbol{\Omega}'_z < 0$ , splitting the regions of integration accordingly and rearranging slightly, we obtain

$$\begin{aligned} & \frac{a}{4\pi} \int_{\mu' < 0} \frac{\mu' L_-(\mathbf{k}, \boldsymbol{\Omega}') - \mu L_-(\mathbf{k}, \boldsymbol{\Omega})}{\mathbf{i}\mathbf{k} \cdot (\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + (\mu - \mu')} d\boldsymbol{\Omega}' \\ & - \frac{a}{4\pi} \mu L_-(\mathbf{k}, \boldsymbol{\Omega}) \int_{\mu > 0} \frac{d\boldsymbol{\Omega}'}{\mathbf{i}\mathbf{k} \cdot (\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + (\mu - \mu')} + L_-(\mathbf{k}, \boldsymbol{\Omega}) \\ & = -\frac{a}{4\pi} \int_{\mu' > 0} \frac{\mu' L_+(\mathbf{k}, \boldsymbol{\Omega})}{\mathbf{i}\mathbf{k} \cdot (\mu\boldsymbol{\omega}' - \mu'\boldsymbol{\omega}) + (\mu - \mu')} d\boldsymbol{\Omega}'. \end{aligned} \quad (12)$$

Because we made the assumption that the refractive indices inside and outside the medium are the same,  $L_+(\mathbf{k}, \boldsymbol{\Omega})$  and  $L_-(\mathbf{k}, \boldsymbol{\Omega})$  are exactly equal to the incident and the reflected radiance, respectively. If this were not the case, we would have had more complicated relations between these quantities [13]. Equation (12) is essentially a linear integral equation, relating the outgoing radiance  $L_-$  to the incoming radiance  $L_+$ . Notice that we do not have to solve for the radiance inside the medium. To obtain the back-reflected radiance, we would have to solve equation (12) for sufficiently many values of  $\mathbf{k}$ ,  $\boldsymbol{\omega}$  and  $\mu$  and then apply a two-dimensional Fourier inversion to the result.

A complication arises from the specific structure of the kernel, because for  $\boldsymbol{\Omega} = \boldsymbol{\Omega}'$  both the numerator and the denominator vanish. The limit is well defined and so the kernel is still integrable, but it is clear that a solution based on straightforward discretization will fail, since that would require the kernel to be evaluated at  $\boldsymbol{\Omega} = \boldsymbol{\Omega}'$ . We will now show how this problem can be dealt with in the simple case where the illuminating beam has infinite width.

Thus, we assume that  $L_0(\mathbf{x}, \boldsymbol{\Omega})$  depends only on  $\boldsymbol{\Omega}$  and because the scattering is isotropic, this dependence is on  $\mu$  only. Hence equation (12) simplifies to

$$\begin{aligned} & \frac{a}{2} \int_{-1}^0 \frac{\mu' L_-(\mu') - \mu L_-(\mu)}{\mu' - \mu} d\mu' - \frac{a}{2} \mu \log \frac{\mu - 1}{\mu} L_-(\mu) - L_-(\mu) \\ & = -\frac{a}{2} \int_0^1 \frac{\mu' L_+(\mu')}{\mu' - \mu} d\mu'. \end{aligned} \quad (13)$$

The same equation, but for the case of no incident radiance, has been treated by Case *et al* [14].

Before continuing, we observe that the criterion that was invoked to derive equation (7) applies equally well to the Fourier transform of the fluence rate, that is, to equation (9). Hence, in order to exclude unwanted ‘runaway solutions’, the numerator in the right-hand side in equation (9) must compensate for the singularity due to the zeros of the denominator, i.e.

$$\int_{4\pi} \frac{\mu' L_0(\mathbf{k}, \boldsymbol{\Omega}')}{\mu' s + \mathbf{i}\mathbf{k} \cdot \boldsymbol{\omega}' + 1} d\boldsymbol{\Omega}' = 0 \quad (14)$$

for that value for  $(s, \mathbf{k})$  for which

$$1 - \frac{a}{4\pi} \int_{4\pi} \frac{d\Omega'}{\mu's + i\mathbf{k} \cdot \boldsymbol{\omega}' + 1} = 0. \quad (15)$$

In the present, one-dimensional formulation, this condition reads

$$\int_{-1}^1 \frac{\mu' L_0(\mu')}{1 + \mu's} d\mu' = 0 \quad (16)$$

for the value of  $s$  for which

$$\frac{a}{2} \int_{-1}^1 \frac{1}{1 + \mu's} d\mu' = 1. \quad (17)$$

This can be integrated to give the well known equation for the ‘critical exponent’ [14]:

$$a \frac{\operatorname{arctanh} s}{s} = 1. \quad (18)$$

Denoting by  $\kappa_0$  the positive solution of equation (18) [14] and substituting this into equation (16), we get

$$\int_{-1}^1 \frac{\mu' L_0(\mu')}{1 + \kappa_0 \mu'} d\mu' = 0. \quad (19)$$

The reason for making this digression lies in the fact that equation (13) does not have a unique solution. This is because the homogeneous equation, i.e. equation (13) without the right-hand side, also possesses nontrivial solutions, which have been calculated by Case *et al* [14]. Physically, these correspond to radiation coming not from the outside, but from sources buried deeply inside the medium. However, by imposing equation (19), we can eliminate these unwanted solutions and make the problem uniquely soluble.

We now turn to the numerical solution of equation (13). To this end we divide the interval  $[-1, 0]$  into  $N$  nodes  $\mu_i$ , with equal distances  $h$  in between and use Simpson’s rule to approximate the kernel. Writing  $f(\mu)$  instead of  $\mu L_-(\mu)$ , we obtain

$$\begin{aligned} \int_{\mu_j}^{\mu_j+2h} \frac{f(\mu) - f(\mu_i)}{\mu - \mu_i} d\mu &\approx \frac{h}{3} \frac{f(\mu_j) - f(\mu_i)}{\mu_j - \mu_i} \\ &+ \frac{4h}{3} \frac{f(\mu_j + h) - f(\mu_i)}{\mu_j + h - \mu_i} + \frac{h}{3} \frac{f(\mu_j + 2h) - f(\mu_i)}{\mu_j + 2h - \mu_i}. \end{aligned} \quad (20)$$

This causes no problems if  $\mu_i$  is not equal to  $\mu_j$ ,  $\mu_j + h$  or  $\mu_j + 2h$ . However, if it is, we must use a technique analogous to ‘product integration’ [15], that is we deduce a quadrature rule for the singular integral.

In fact, we assume that the integral can be approximated by

$$\int_{\mu_j}^{\mu_j+2h} \frac{f(\mu) - f(\mu_i)}{\mu - \mu_i} d\mu \approx w_1 f(\mu_j) + w_2 f(\mu_j + h) + w_3 f(\mu_j + 2h) \quad (21)$$

and determine the  $w_i$  such that the approximate equality becomes an exact one if  $f(\mu)$  is a polynomial of at most degree two. For example, if  $\mu_i = \mu_j$ , and if we set  $f(\mu)$  equal to 1,  $\mu - \mu_i$  and  $(\mu - \mu_i)^2$ , respectively, we obtain from (21)

$$w_1 + w_2 + w_3 = 0 \quad (22)$$

$$hw_2 + 2hw_3 = 2h \quad (23)$$

$$h^2w_2 + 4h^2w_3 = 2h^2. \quad (24)$$

This can be solved to give  $w_1 = -2, w_2 = 2, w_3 = 0$ . Thus, if  $\mu_i = \mu_j$ , we have the quadrature rule

$$\int_{\mu_j}^{\mu_j+2h} \frac{f(\mu) - f(\mu_i)}{\mu - \mu_i} d\mu \approx -2f(\mu_j) + 2f(\mu_j + h). \quad (25)$$

In a similar fashion we find that  $(w_1, w_2, w_3) = (-1, 0, 1)$  and  $(w_1, w_2, w_3) = (-2, 2, 0)$ , if  $\mu_i = \mu_j + h$  and  $\mu_i = \mu_j + 2h$ , respectively.

Thus, using the quadrature rules pointed out in equations (20) and (25), we can obtain a linear system of equations in the unknown  $f(\mu_i)$ . (Actually, because  $f(\mu) = \mu L_-(\mu)$ ,  $f(0)$  is always zero.) We may write this system as one matrix equation, say

$$\mathbf{A}\mathbf{f} = \mathbf{b} \quad (26)$$

where  $\mathbf{A}$  is the coefficient matrix,  $\mathbf{f} = [f(\mu_1), \dots, f(\mu_N)]$  is the unknown vector and  $\mathbf{b}$  is the known vector. Unfortunately, this system is ill-conditioned. This is not surprising in as much as the integral equation (13) from which it is derived is itself singular. Although treating equation (26) in a straightforward manner will, in general, produce a solution, this will be unstable [16] and it will not satisfy the extra requirement, given by equation (19). This means that part of the solution will actually be due to radiation from deeply buried sources, instead of from outside the medium. It turns out that we can fix either problem by incorporating the above-mentioned requirement (19) through the method of Lagrange multipliers.

To this end, we start by converting equation (19) into a discrete form. Splitting the integral into the incoming part ( $\mu > 0$ ) and the outgoing part ( $\mu < 0$ ) and approximating the latter by Simpson's rule, we obtain

$$\sum_{i=1}^N u_i \frac{f(\mu_i)}{1 + \kappa_0 \mu_i} = - \int_0^1 \frac{\mu L_+(\mu)}{1 + \kappa_0 \mu} d\mu \quad (27)$$

where  $u_i$  denote the weights from Simpson's rule. Setting  $v_i = u_i / (1 + \kappa_0 \mu_i)$ , we can write equation (27) in vector notation as

$$\mathbf{v}^T \mathbf{f} = - \int_0^1 \frac{\mu L_+(\mu)}{1 + \kappa_0 \mu} d\mu. \quad (28)$$

Now, setting  $\mathbf{r} = \mathbf{A}\mathbf{f} - \mathbf{b}$ , we look for the vector  $\mathbf{f}$  which minimizes  $\mathbf{r}^T \mathbf{r}$ , subject to (28). From the Lagrange multiplier method, we know that the vector we look for satisfies

$$\nabla_{\mathbf{f}}(\mathbf{r}^T \mathbf{r}) + 2\lambda \nabla_{\mathbf{f}}(\mathbf{v}^T \mathbf{f}) = 0 \quad (29)$$

where the parameter  $\lambda$  is yet to be determined. The gradients are easily calculated. In fact, we have

$$\begin{aligned} \nabla_{\mathbf{f}}(\mathbf{r}^T \mathbf{r}) &= \nabla_{\mathbf{f}}(\mathbf{A}\mathbf{f} - \mathbf{b})^T (\mathbf{A}\mathbf{f} - \mathbf{b}) \\ &= \nabla_{\mathbf{f}}(\mathbf{f}^T \mathbf{A}^T \mathbf{A}\mathbf{f} - \mathbf{f}^T \mathbf{A}^T \mathbf{b} - \mathbf{b}^T \mathbf{A}\mathbf{f} + \mathbf{b}^T \mathbf{b}) \\ &= 2\mathbf{A}^T \mathbf{A}\mathbf{f} - 2\mathbf{A}^T \mathbf{b} \end{aligned} \quad (30)$$

and

$$\nabla_{\mathbf{f}}(\mathbf{v}^T \mathbf{f}) = \mathbf{v}. \quad (31)$$

Hence, from (29) it follows that the sought after solution obeys

$$\mathbf{A}^T \mathbf{A}\mathbf{f} + \lambda \mathbf{v} = \mathbf{A}^T \mathbf{b}. \quad (32)$$

Together with the constraint (28), equation (32) constitutes a system of  $N$  (without  $f(0)$ , but including  $\lambda$ ) linear equations with an equal number of variables. This system is no longer

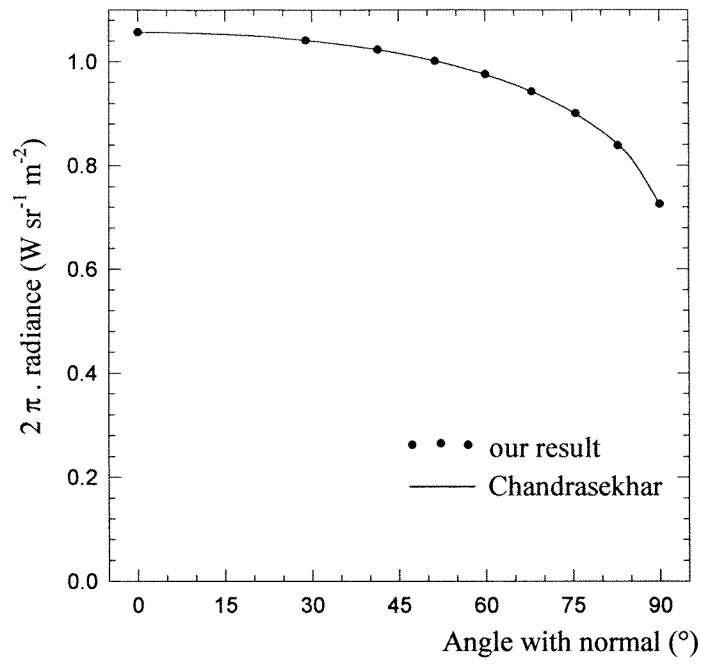


Figure 1. Back-reflected radiance.  $a = 1$ , normally incident beam.

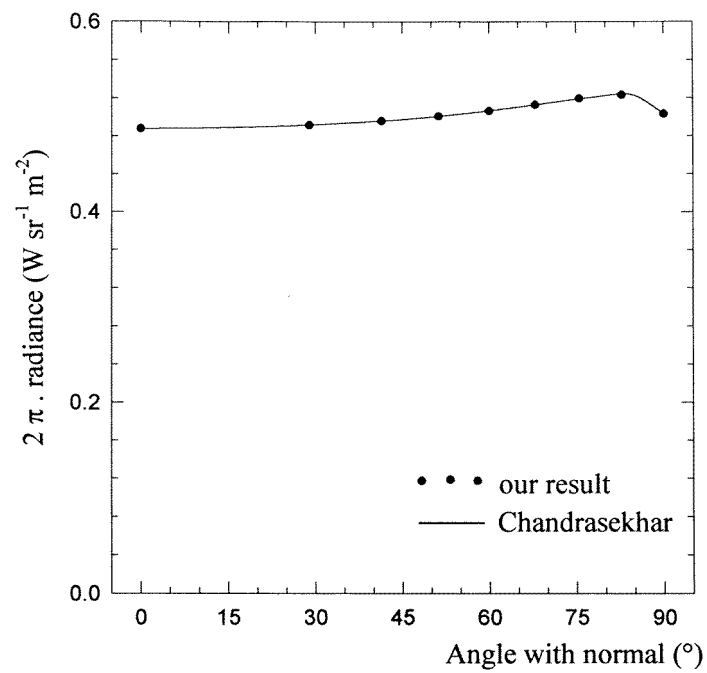


Figure 2. Back-reflected radiance.  $a = 1$ , beam incident at  $60^\circ$  with respect to the normal.

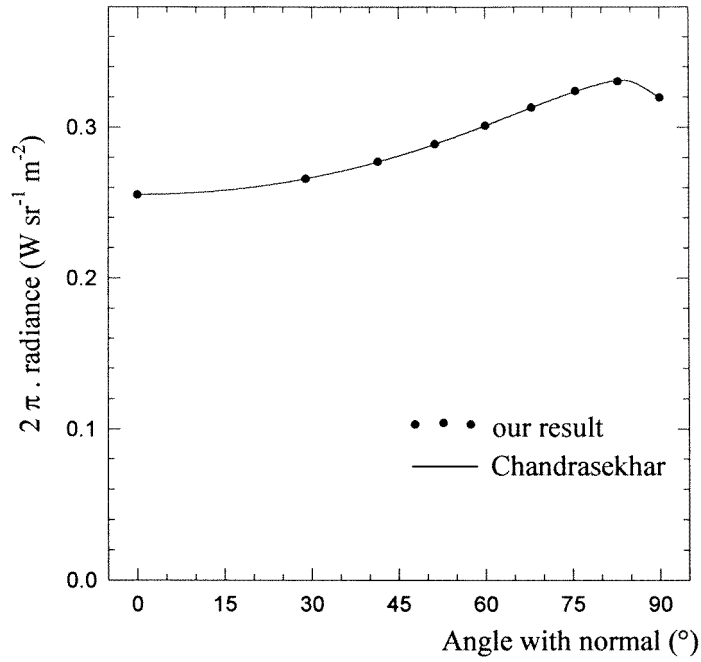


Figure 3. Back-reflected radiance.  $a = 0.8$ , normally incident beam.

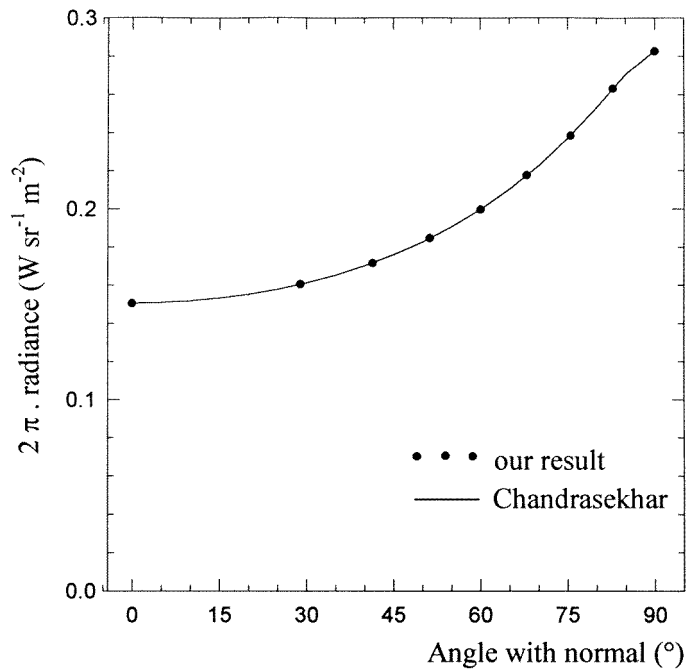


Figure 4. Back-reflected radiance.  $a = 0.8$ , beam incident at 60° with respect to the normal.



ill-conditioned and can be solved in a straightforward manner. To obtain the radiance, which we were actually interested in, we simply divide  $f(\mu_i)$  by  $\mu_i$ . Of course, for  $\mu = 0$ , this is not possible, but then we have from equation (13)

$$\frac{1}{2}a \int_{-1}^0 L_{-}(\mu') d\mu' - L_{-}(0) = -\frac{1}{2}a \int_0^1 L_{+}(\mu') d\mu'. \quad (33)$$

If once again we approximate the integral by Simpson's rule and use the already calculated values of  $L_{-}(\mu)$  for  $\mu \neq 0$ , it is an easy matter to determine  $L_{-}(0)$  from this last equation.

The case of conservative scattering (i.e. the case where  $a = 1$ ) often needs special treatment [17], but for the present method this is not so. Examples are given in the next section.

### 3. Results

We calculated back-reflected radiances due to a collimated, infinitely wide beam of  $1 \text{ W m}^{-2}$ , incident normally and at a  $60^\circ$  angle onto the surface of an isotropically scattering half-space of albedo 1 and 0.8, respectively. Figures 1–4 show the result for an eight-point discretization. The mesh points were chosen so as to make  $\cos \theta$  equidistant, so on a linear  $\theta$  scale they are dense for  $\theta$  close to  $90^\circ$  and sparse for  $\theta$  close to zero.

Although the equations that we deduced have not been presented in quite the same way, the same calculations have been done by other authors, using other, and sometimes rather involved, methods [17, 18]. This permits a comparison of our results with those of, for example, [18]. In the figures, our results are compared with those obtained by Chandrasekhar [18] (full curves). It turns out that the differences are of the order of  $10^{-4}$ , which is roughly the error in Chandrasekhar's results.

### 4. Conclusions

In this paper we have made some simplifications in order to present the basics of the technique as clearly as possible. Therefore we have assumed an infinitely wide homogeneous beam, because this makes the problem one dimensional. We showed that the back-reflected beam could be expressed directly in terms of the incident beam, *without one having to solve the radiative transfer equation in the scattering medium*. In practice we will have to deal with the more complicated situation of a finite beam and/or anisotropic scattering.

Let us first assume isotropic scattering, but a beam of finite width. In that case, we can still use equation (12), since this equation is valid for any incoming field, in particular for a finite beam. The spatial extent of the beam is expressed by the variables  $\mathbf{k}$  through its Fourier transform. However,  $\mathbf{k}$  is a spectator variable. This means that we can carry out the calculation for sufficiently many values of  $\mathbf{k}$  and use the technique presented in this paper. Afterwards, the result has to be transformed to real space, which requires more computational effort than does the simple case treated here.

The case of *anisotropic* scattering becomes more involved. In the present paper we have used the moment  $\int L(\mathbf{x}, \boldsymbol{\Omega}) d\boldsymbol{\Omega}$  in order to express the back-reflected radiance in terms of the incident radiance. For anisotropic scattering, other moments have to be considered as well, the exact number depending on the scattering phase function. Expanding  $L(\mathbf{x}, \boldsymbol{\Omega})$  in terms of the spherical harmonics, one can derive a difference-differential equation for the expansion coefficients. We believe that the technique presented in this paper can also be extended to this regime, but the treatment will become more complicated.

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